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Breaking and merging of liquid sheets and filaments

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Abstract. The surface of a sheet of liquid which contracts due to surface tension, breaks, and then pulls apart into two pieces, is calculated. Before breaking, the flow is the self-similar one found by Keller, Milewski and Vanden-Broeck. After breaking, it is the self-similar flow found by Keller and Miksis. A general numerical scheme, which includes the previous ones, is presented and new numerical results are discussed. There is an analogous flow of an axially symmetric liquid filament, but it is not calculated.

Key words: boundary-integral equations, free-surface flows, self-similar solutions, surface tension.

1. Introduction

Whenever a liquid sheet or filament breaks, or two bodies of fluid merge, the domain containing the liquid changes its topology discontinuously. For example, it may change from being connected to being disconnected, as happens when a filament breaks, or from being simply connected to being multiply connected, as occurs when a hole forms in a sheet. Accompanying this topological discontinuity is a singularity in the fluid velocity at the instant of breaking or merging. The singular behavior of the topology and of the velocity make it difficult to analyze breaking and merging flows. Therefore it is of interest to find examples of such flows which can be analyzed, so we shall present one in two dimensions and a similar one in three dimensions.

Our examples involve the potential flow of an inviscid incompressible liquid of density ρ acted upon by surface tension σ. The flow is reversible, so the breaking flow, when reversed, yields a merging flow. Therefore we shall just describe the breaking flow. At the instant of breaking, $t = 0$, a two dimensional sheet of liquid is bounded by two planes which intersect along the z-axis, as shown in Figure 1b. The liquid is at rest and occupies two wedge-shaped regions with their edges in contact. In the three dimensional case, a liquid filament is bounded initially by two nappes of a cone. The liquid is at rest and occupies two conical regions with their vertices in contact at the origin.

Keller and Miksis [1] showed by dimensional analysis that in both of these cases the flow and the boundary of the fluid domain are self-similar for $t > 0$. In both cases, the equation for the boundary is of the form $F\left[\mathbf{x}/(t^2\sigma/\rho)^{1/3}\right] = 0$, and the velocity potential is of the form $t^{1/3}(\sigma/\rho)^{2/3}\varphi\left[\mathbf{x}/(t^2\sigma/\rho)^{1/3}\right]$. Thus lengths grow like $t^{2/3}$ as t increases. The velocity decays like $t^{-1/3}$ at a fixed value of $\mathbf{x}/(t^2\sigma/\rho)^{1/3}$, and at any fixed value of $\mathbf{x} \neq 0$ it tends to zero at $t = 0$. For breaking, the boundary was determined numerically in [1, Figure 2] for the two dimensional case, for various values of the wedge angle. The fluid is contained in two domains shown in Figure 1c, which continually separate from one another.

The same analysis applies for $t < 0$, before breaking, when the fluid occupies one domain. This case, shown in Figure 1a, was solved numerically by Keller *et al.* [2, Figure 2], who considered the merging of two wedges of fluid initially at rest with their edges in contact. Since the flow is reversible, their solution can be reversed to represent the gradual thinning of a liquid sheet which, at $t = 0$, becomes two wedges of fluid at rest with their edges in contact.

We now use the time reversal of the merging flow obtained in [2] followed by the retracting or separating flow obtained in [1]. This yields a flow which represents the gradual thinning of a single sheet of fluid (Figure 1a). It reaches zero thickness along the line $z = 0$ at $t = 0$ (Figure 1b). Then it splits apart into two regions of fluid which separate rapidly from one another (Figure 1c). Figure 1 applies for a 45-degree slope angle of the free surface far out in the first quadrant. Solutions like these can be obtained for other slope angles.

The flow just described is very special, which is why it can be analyzed. Exactly the same description applies to the filament in three dimensions, but the solutions have not been calculated numerically.

Previously, Eggers [3] used two solutions to describe the shape and flow of an axially symmetric filament of viscous liquid, one valid before pinch-off and one valid after. He considered the first term in the expansion of the velocity in powers of the radial variable, which depended upon z and t. He found two similarity solutions which satisfied an ordinary differential equation. Day *et al.* [4] solved numerically for the flow before pinch-off of an axially symmetric drop which was initially dumbbell shaped. They found that for a variety of initial conditions, its shape and velocity tended to a self-similar form near the breaking point.

The discussion above shows that the breaking problem in [1] is related to the merging problem in [2]. This suggests combining the numerical procedures in [1] and [2] into a unified scheme. This is achieved in Section 2. In Section 3, new numerical solutions are presented.

2. Formulation

We consider two wedges of fluid touching at their vertices (see Figure 2a). We introduce cartesian coordinates with the origin at O. The configuration is assumed to be symmetric with respect to the x-axis. Only the portion of the configuration in $y > 0$ is shown in Figure 2a. The lines OA and OB make angles θ_1 and θ_2 with the positive x-axis.

At $t = 0$, the fluid begins to move under the influence of surface tension. There are two possible scenarios. In the first the fluid splits apart into two regions of fluid (see Figure 2b). In the second the two wedges merge into a single fluid region (see Figure 2c). These two problems have been studied by Keller and Miksis [1] and by Keller *et al.* [2]. We shall refer to them as the breaking problem (Figure 2b) and the merging problem (Figure 2c). Since the initial configuration is symmetric with respect to the x-axis, the resulting flow will be symmetric also. Therefore we only show in Figures 2b and 2c the flow in the region $y > 0$.

The fluid velocity can be written as $\nabla\Phi(\mathbf{x}, t)$ where the potential Φ is a harmonic function in the fluid domain $\Omega(t)$:

$$\Delta\Phi = 0, \qquad \mathbf{x} \in \Omega(t). \tag{2.1}$$

For the merging problem, $\Omega(t)$ is the 'FLUID' area of Figure 2c. For the breaking problem, the two disjoint 'FLUID' areas of Figure 2b can be calculated independently. Therefore we assume first, that $\Omega(t)$ is the 'FLUID' area $x > 0$, $y > 0$ of Figure 2b.

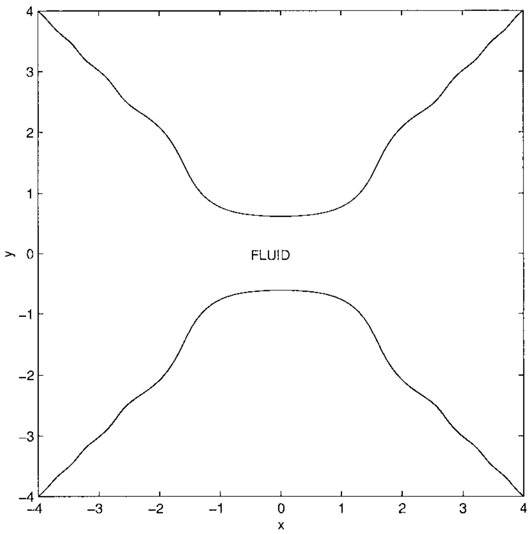

Figure 1a. The self-similar surfaces of a sheet of liquid are shown for $t < 0$. The sheet is becoming thinner near the origin as the liquid flows outward. The surfaces are symmetric about the x and y axes. In the variables $x/(t^2\sigma/\rho)^{1/3}$, $y/(t^2\sigma/\rho)^{1/3}$ the surfaces are independent of t for $t < 0$.

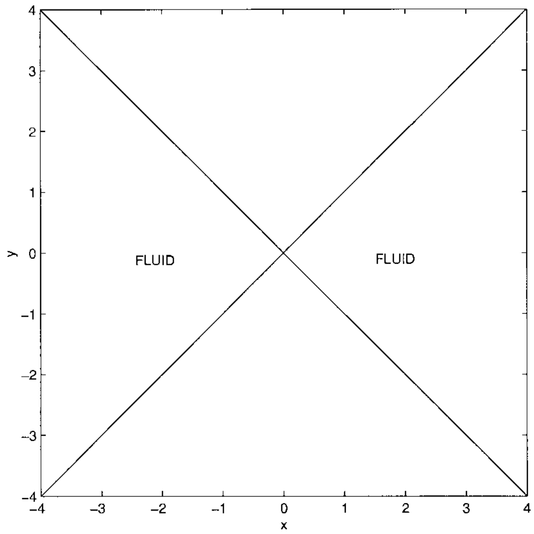

Figure 1b. The limit at $t = 0$ of the sheet shown in Figure 1a. The limit of the fluid velocity is zero at every point other than the origin. The surfaces of the fluid are the straight lines $y = \pm x$. This is the configuration at the instant of breaking.

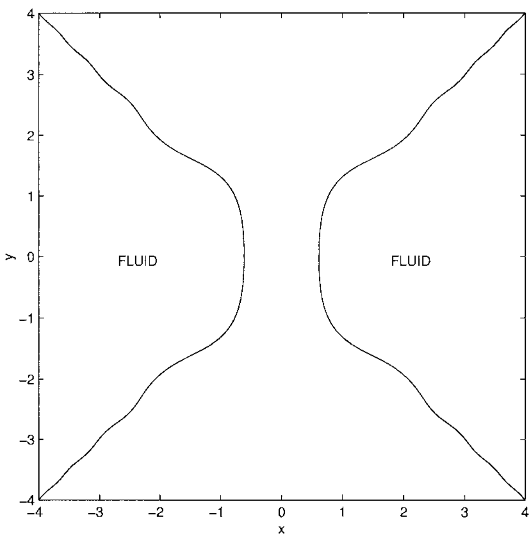

Figure 1c. For $t > 0$, the self-similar surfaces of the two parts into which the sheet has broken are shown. The surfaces tend to the configuration shown in Figure 1b as $t \to 0$, and the velocity tends to zero as $t \to 0$ at every point except the origin. In the variables $x/(t^2\sigma/\rho)^{1/3}$, $y/(t^2\sigma/\rho)^{1/3}$ the surfaces are independent of t for $t > 0$.

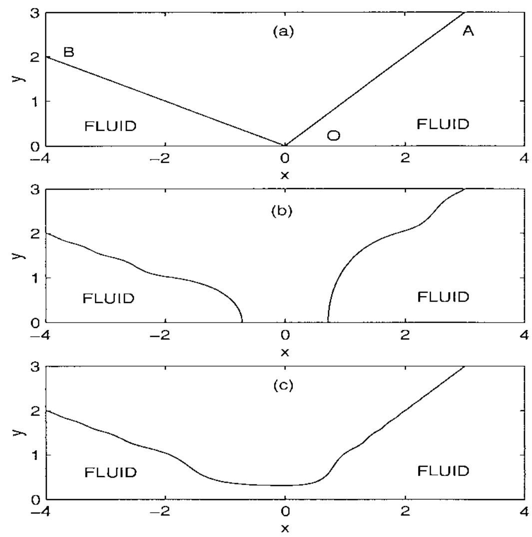

Figure 2. Sketches of the flow and of the coordinates. Each of the three flows is assumed to be symmetric with respect to the x-axis. Only the portions in $y > 0$ are shown.

For simplicity we shall first describe the formulation of the merging problem and indicate at the end of Section 3 the required changes for the breaking problem. The free boundary of $\Omega(t)$ is described by $F(\mathbf{x}, t) = 0$. On this boundary, the kinematic and dynamic boundary conditions are, respectively,

$$F_t + \nabla\Phi \cdot \nabla F = 0, \quad \text{on} \quad F = 0, \tag{2.2}$$

$$\Phi_t + \frac{1}{2}(\nabla\Phi)^2 + (\sigma/\rho)\kappa = 0 \quad \text{on} \quad F = 0. \tag{2.3}$$

Here, σ is the coefficient of surface tension, ρ is the density of the fluid, and κ is the curvature of the free surface. The symmetry about the x-axis implies

$$\Phi_y = 0 \quad \text{on} \quad y = 0. \tag{2.4}$$

To fix the velocity potential uniquely, we assume that

$$\Phi \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \tag{2.5}$$

Initially, at $t = 0$, we require that

$$\Phi(\mathbf{x}, 0) = 0, \tag{2.6}$$

$$F(\mathbf{x}, 0) = y \cos\theta_1 - x \sin\theta_1, \quad x > 0, \tag{2.7}$$

$$F(\mathbf{x}, 0) = y \cos\theta_2 - x \sin\theta_2, \quad x < 0. \tag{2.8}$$

This concludes the formulation of the problem. For given values of θ_1 and θ_2, we seek Φ and the curve $F = 0$ such that (2.1)–(2.8) are satisfied.

Since this problem contains no length scale, we shall seek a solution expressed in terms of the dimensionless similarity variables ξ and η defined by

$$\xi = x(\rho/\sigma t^2)^{1/3}, \qquad \eta = y(\rho/\sigma t^2)^{1/3}. \tag{2.9}$$

We write Φ and F in terms of two new functions φ and f:

$$\Phi(x, y, t) = (\sigma^2 t/\rho^2)^{1/3}\varphi(\xi, \eta), \tag{2.10}$$

$$F(x, y, t) = f(\xi, \eta). \tag{2.11}$$

The domain $\Omega(t)$ becomes a fixed domain Ω_0 in the ξ, η variables, with the equation of the free boundary given by $f(\xi, \eta) = 0$. Substituting (2.9)–(2.11) into (2.1)–(2.3) yields,

$$\Delta\varphi = 0 \quad \text{in} \quad \Omega_0, \tag{2.12}$$

$$-\tfrac{2}{3}(\xi, \eta) \cdot \hat{n} + \nabla\varphi \cdot \hat{n} = 0 \quad \text{on} \quad f = 0, \tag{2.13}$$

$$-\tfrac{2}{3}(\xi, \eta) \cdot \nabla\varphi + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{3}\varphi + \kappa = 0 \quad \text{on} \quad f = 0. \tag{2.14}$$

In (2.13), $\hat{n}$ denotes the unit normal pointing out of Ω_0. Equations (2.4) and (2.5) remain unchanged with Φ, x, y, F replaced by φ, ξ, η, f respectively. Equation (2.6) follows from (2.10), while (2.7), (2.8) and (2.11) yield

$$f(\xi, \eta) \sim \eta \cos\theta_1 - \xi \sin\theta_1 \quad \text{as} \quad \xi^2 + \eta^2 \to \infty \quad \text{with } \xi > 0, \tag{2.15}$$

$$f(\xi, \eta) \sim \eta \cos\theta_2 - \xi \sin\theta_2 \quad \text{as} \quad \xi^2 + \eta^2 \to \infty \quad \text{with } \xi < 0. \tag{2.16}$$

3. Integro-differential formulation

To solve the problem formulated in Section 2, we first express the free surface $f(\xi, \eta) = 0$ parametrically in terms of arclength s along it: $\xi = \bar{\xi}(s)$, $\eta = \bar{\eta}(s)$. Then

$$\bar{\xi}'^2 + \bar{\eta}'^2 = 1. \tag{3.1}$$

The tangent $\hat{t}$ and the normal $\hat{n}$ are given by

$$\hat{t} = (\bar{\xi}', \bar{\eta}'), \qquad \hat{n} = (-\bar{\eta}', \bar{\xi}'), \tag{3.2}$$

and the curvature κ is

$$\kappa = \bar{\eta}'\bar{\xi}'' - \bar{\xi}'\bar{\eta}''. \tag{3.3}$$

On the boundary the gradient of φ can be written as

$$\nabla\varphi = \hat{n}\frac{\partial\varphi}{\partial n} + \hat{t}\frac{\partial\varphi}{\partial s}. \tag{3.4}$$

By using these relations, we can write the boundary conditions (2.13) and (2.14) as

$$\frac{\partial\varphi}{\partial n} = \frac{2}{3}(\bar{\eta}\bar{\xi}' - \bar{\xi}\bar{\eta}'), \tag{3.5}$$

$$-\frac{2}{3}(\bar{\xi}\bar{\xi}' + \bar{\eta}\bar{\eta}')\frac{\partial\varphi}{\partial s} + \frac{1}{2}\left(\frac{\partial\varphi}{\partial s}\right)^2 - \frac{1}{2}\left(\frac{\partial\varphi}{\partial n}\right)^2 + \frac{1}{3}\varphi + \bar{\eta}'\bar{\xi}'' - \bar{\xi}'\bar{\eta}'' = 0. \tag{3.6}$$

On the free surface, the far field conditions (2.5), (2.15) and (2.16) become, respectively,

$$\varphi \to 0 \quad \text{as} \quad s \to \infty, \tag{3.7}$$

$$\bar{\eta} \cos\theta_1 - \bar{\xi} \sin\theta_1 = 0 \quad \text{as} \quad s \to +\infty \tag{3.8}$$

$$\bar{\eta} \cos\theta_2 - \bar{\xi} \sin\theta_2 = 0 \quad \text{as} \quad s \to -\infty. \tag{3.9}$$

We introduce the Green's function $G(\xi, \eta, \xi^*, \eta^*)$ satisfying

$$\Delta G = \delta(\xi - \xi^*, \eta - \eta^*), \tag{3.10}$$

$$\frac{\partial G}{\partial \eta} = 0, \quad \text{on} \quad \eta = 0. \tag{3.11}$$

Using the method of images, we find that G is given by

$$G = \frac{1}{2\pi}\log[(\xi - \xi^*)^2 + (\eta - \eta^*)^2]^{1/2} + \frac{1}{2\pi}\log[(\xi - \xi^*)^2 + (\eta + \eta^*)^2]^{1/2} \tag{3.12}$$

We now apply Green's theorem in Ω_0 to φ and G, writing $G(s, s^*) = G[\bar{\xi}(s), \bar{\eta}(s), \bar{\xi}(s^*), \bar{\eta}(s^*)]$ and $\varphi(s) = \varphi[\bar{\xi}(s), \bar{\eta}(s)]$. The only nonzero contribution to the integrals comes from the free surface, so we get

$$\frac{1}{2}\varphi(s^*) = \int_{-\infty}^{\infty} \left[\varphi(s) \frac{\partial G}{\partial n}(s, s^*) - G(s, s^*) \frac{\partial \varphi}{\partial n}(s) \right] \mathrm{d}s. \tag{3.13}$$

Equation (3.13), together with the conditions (3.1) and (3.5)–(3.9) define an integro–differential system for the unknowns $\bar{\xi}(s)$, $\bar{\eta}(s)$, $\varphi(s)$. This concludes the formulation of the merging problem.

For the breaking problem, $\Omega(t)$ is the 'FLUID' area $x > 0$, $y > 0$ of Figure 2b. Therefore the integral from $-\infty$ to ∞ in (3.13) is replaced by an integral from 0 to ∞. This new equation (3.13), together with the conditions (3.5)–(3.8) define the integro-differential equation for the 'FLUID' in the region $x > 0$ of Figure 2b. The corresponding problem for the 'FLUID' area in the region $x < 0$ of Figure 2b is obtained by solving the same equations with θ_1 replaced by $\pi - \theta_2$ in (3.8). At the end of the calculations the flow is refected across the y-axis.

These various integro–differential equations can be solved by using the finite difference methods described in [1] and [2]. The reader is referred to these papers for details.

4. Discussion of the numerical results

The numerical scheme described in Section 3 was used to compute solutions for various values of θ_1 and θ_2.

We first consider solutions which are symmetric with respect to the y-axis (*i.e.* we assume $\theta_2 = \pi - \theta_1$) Typical solutions for the breaking and merging problems are shown in Figure 1. Here $\theta_1 = \pi/4$. One feature of these solutions is the presence of capillary waves. These waves are of decaying amplitude as $|s| \to \infty$. Keller and Miksis ([1]) derived an asymptotic solution which agrees with the numerical results for $|s|$ large. As θ_1 increases, the amplitude of the capillary waves increases. This is illustrated in Figure 2 of [2] where profiles are presented for various values of $\theta_1 < 85°$. We note that the profiles in [2] need to be rotated by 90° to give those presented here. These results suggest that the free surface will ultimately come into contact with itself for θ_1 near 90°. To check this idea we calculated solutions for $\theta > 85°$. These computations show that the free surface develops a point of contact with itself at $\theta_1 \approx 87.5°$, trapping a 'bubble'. The corresponding free surface profiles are shown in Figure 3. We note that since the solutions are self-similar, the area of the trapped bubble is not conserved, and therefore trapping cannot occur in a configuration of two incompressible fluids. For the same reason, pinchoff cannot occur in the self similar solution for either part in the breaking of a single incompressible fluid.

There are different ways to interpret the solutions in Figure 1. In [1] and [2], the initial configuration is the flow of Figure 1b. The calculations in [1] are concerned with the breaking problem in which the flow evolves for $t > 0$ into the configuration of Figure 1c. Those in [2] describe the merging problem in which the flow evolves into the configuration of Figure 1a. As described in the introduction, there is a third interpretation which is obtained by reversing the flow in [2]. We then have a gradual thinning of a single sheet (Figure 1a). It reaches zero thickness at $t = 0$ (Figure 1b) and then splits into two regions (Figure 1c).

So far all the solutions presented are symmetric with respect to the y-axis. In Figure 4 we present a merging solution for $\theta_1 = 50°$ and various values of θ_2. If we use the third

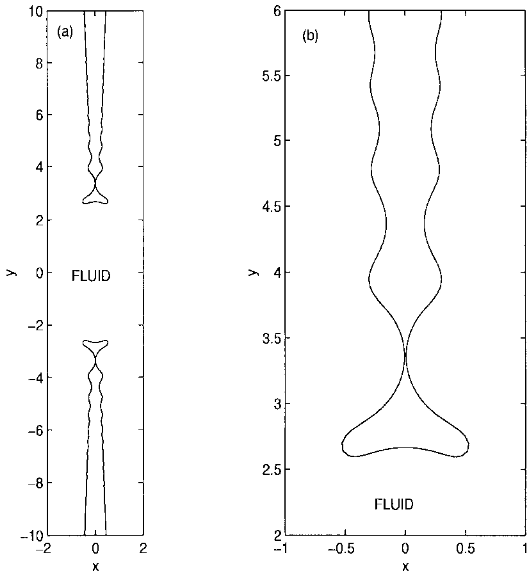

Figure 3. (a) Merging flow for $\theta_1 = 87{\cdot}5°$. The flow is symmetric with respect to $x = 0$. There are trapped bubbles at the tips of the free surface. (b) Enlarged profile of a trapped bubble.

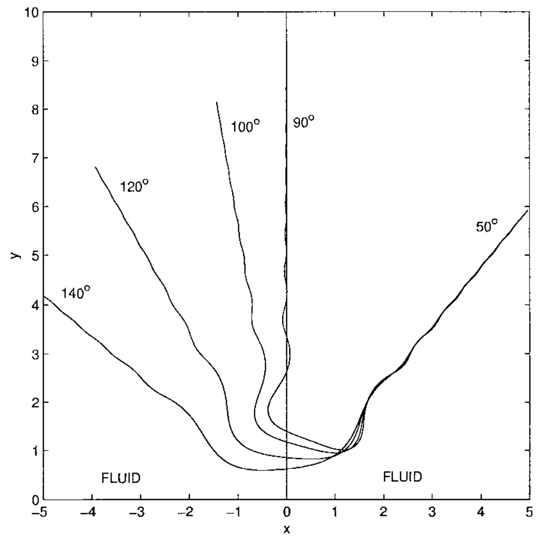

Figure 4. Nonsymmetric merging flows with $\theta_1 = 50^0$ and various values of θ_2.

interpretation, this sheet will gradually thin, reach zero thickness and then evolve into two separate regions symmetric with respect to the x-axis.

5. Conclusions

We have presented a numerical procedure which unifies the results in [1] and [2]. We have shown that the solutions in [1] and [2] can be used to describe the progressive thinning and breaking of a sheet of liquid. We have presented new numerical solutions showing that there are solutions for which the free surface touches itself at one point and encloses a small bubble. Nonsymmetric merging flows were also presented. There also are self similar solutions describing fluids in contact with rigid walls. These flows are calculated in [1] and [2].

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